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Determination and Study of Positive Forms on Spaces of Functions, II*

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III. SUBMEASURES

In what follows E will be a substonian space, $\mathscr{M}^+(E)$ its set of positive Radon measures with the vague topology, and V a linear subspace of $\mathscr{D}(E)$ positively generated, i.e., $V = V^+ - V^+$, where $V^+ = V \cap \mathscr{D}^+(E)$. The convex cone of positive linear forms on V will be denoted $V_*{}^+$.

For any couple (f, μ), where $f \in \mathscr{D}^+(E)$ and $\mu \in \mathscr{M}^+(E)$ with $S(\mu) \subset S(f)$, such that for any $g \in V$, the subquotient g/f is μ-integrable, the mapping $g \to \mu(g/f)$ is a positive form on V which will be denoted $[f, \mu]$.

DEFINITION 16. An element $T \in V_*{}^+$ is called a *submeasure*[1] if it is equal to some $[f, \mu]$; it is called a *proper submeasure* if this is possible with $f \in V^+$, and a *measure* if $T = [1, \mu]$.

T is called a *subvaluation* if it is equal to some $[f, \epsilon_a]$, where ϵ_a is the Dirac measure supported by $a \in E$; the point a is then called a *pole* of this subvaluation (it is not always unique).

T is *a valuation* if it is equal to some $[1, \epsilon_a]$, i.e., if $T(g) = g(a)$ for any $g \in V$.

Remarks 17. (i) Obviously $[f, \mu] = [2f, 2\mu]$ so that the representation of a submeasure by $[f, \mu]$ is never unique.

(ii) If $[f, \mu]$ is a submeasure, then for any positive measure $\mu' \leqslant \mu$, and any $f' \in \mathscr{D}^+(E)$ with $f \leqslant f'$, $[f', \mu']$ is also a submeasure, with $[f', \mu'] \leqslant [f, \mu]$.

EXAMPLES 18.

(i) Let B be the closed unit ball of Euclidian space $\mathbb{R}^n$; let U be the linear space of 1-Lipschitz functions on B which are identically 0 on the

* Continuation of part I, this Journal **7** (1973), 325–333.

[1] The choice of "submeasure" instead of "pseudomeasure" used originally in Choquet [1] is justified by "substonian" and also to avoid confusion with pseudomeasures used sometimes in harmonic analysis.

boundary B^* of B; and finally let φ be the element of U defined by $\varphi(x) =$ (distance of x to B^*). Let E be the stonian space of all ultrafilters on B; let V be the linear space of continuous extensions of elements of U to E; and finally let f be the continuous extension of φ to E.

To any $\mu \in \mathcal{M}^+(E)$ corresponds the submeasure $[\varphi, \mu]$ on V. When μ is supported by F, the closed set of ultrafilters on $\mathring{B}$ converging to some point of B^*, the submeasure $[f, \mu]$ cannot be represented as a measure $[1, \nu]$.

(ii) In an analogous way, the positive forms T_1, T_2 studied in the Introduction, could be interpreted in terms of submeasures.

(iii) Now, here is a positive form which is *not* a submeasure: Let E be a substonian space containing some *nondenumerable* discrete open set 0 (for instance $E = \beta I$ with I nondenumerable), and let π be the Radon measure on the subspace 0 of E such that $\pi(\{x\}) = 1$ for any $x \in 0$. Let V be the space of all $g \in \mathscr{C}(E)$ such that $\pi(|g|) < \infty$; then the mapping $g \to \pi(g)$ is a positive form on V which is not a submeasure.

We will study now several operations on submeasures.

PROPOSITION 19. *Let $[f, \mu]$ be a submeasure on V. For any $f' \in \mathscr{D}^+(E)$ with $f \leqslant f'$, $[f', \mu] = [f, \mu']$, with $\mu' = (f/f')\mu$.*

Proof. Note that for any $g \in V$, $g/f' = (f/f')(g/f)$ on $S(f)$, hence $\mu(g/f') = \mu((f/f') \cdot (g/f))$, or in other words $\mu'(g/f)$, where μ' is the product of μ by f/f', which is continuous on $S(f)$.

PROPOSITION 20.

(i) *For any* proper *submeasure $[f, \mu]$ on V and for any $f' \in V^+$ with $f \leqslant f'$, we have $[f, \mu] = [f', \mu']$, where μ' is given by the product $\mu' = (f'/f)\mu$.*

(ii) *For any finite family of* proper *submeasures $[f_i, \mu_i]$, its sum is the proper submeasure $[f, \mu]$ where $f = \sum f_i$ and $\mu = \sum (f/f_i)\, \mu_i$,*

Proof.

(i) For any $g \in V^+$, $[f, \mu](g) = \mu(g/f)$. Now on $S(f)$, $g/f = ab$, where $a = f'/f$ and $b = g/f'$. As $f' \in V^+$ and $f \leqslant f'$, a, b and ab are μ-integrable, so that $\mu(ab) = a\mu(b)$, which is the relation we want to prove.

(ii) As $f \in V^+$, (i) shows that $[f_i, \mu_i] = [f, \mu_i']$, where μ_i' is the product $(f/f_i)\, \mu_i$; hence the relation.

Remark 21. The proof of Property 20(i) is based on the fact that $f' \in V^+$, and would not be valid for an arbitrary $f' \in \mathscr{D}^+(E)$, even with $f \leqslant f'$. However it is true that for any $f, f' \in \mathscr{D}^+(E)$ with $S(f) \subset S(f')$ and any submeasure $[f, \mu]$ such that f/f' is μ-integrable and $\neq 0$ μ-almost everywhere, the following holds:

Each g/f' with $g \in V^+$, is μ'-integrable, where $\mu' = (f'/f)\mu$; and $[f, \mu] = [f', \mu']$.

The proof is the same as for 20(i).

Remark 22. The proof of 20(ii) cannot be extended to functions f_i not in V^+; it is not known if the property is still valid in that case.

We want now to define a useful notion. Let I be a directed set, $i \to f_i$ an increasing mapping of I into $\mathscr{D}^+(E)$, and $i \to \mu_i$ a mapping of I into $\mathscr{M}^+(E)$ such that $S(\mu_i) \subset S(f_i)$ for every i.

DEFINITION 23. The family of those couples (f_i, μ_i) is called *projective* whenever $(i \leqslant j) \Rightarrow (\mu_i = (f_i/f_j)\, \mu_j)$.

EXAMPLE 24. Let us suppose that V is *hereditary*, i.e., $(f \in V^+, g \in \mathscr{D}^+(E), g \leqslant f) \Rightarrow (g \in V^+)$.

Then $(f \in V$ and $\varphi \in \mathscr{C}(E)) \Rightarrow (\varphi f \in V)$ so that for any $T \in V_*^+$, and $f \in V^+$, the mapping $\varphi \to T(\varphi f)$ is a positive form on $\mathscr{C}(E)$, that is to say an element of $\mathscr{M}^+(E)$, which we will denote $\{T, f\}$. The mapping $(T, f) \to \{T, f\}$ is obviously bilinear.

Let us suppose now that T is fixed; as V^+ is a directed set, it has a meaning to investigate whether the mapping $f \to (f, \{T, f\})$ is projective; let us prove that it is indeed the case. It is obviously increasing and for every f, $\{T, f\}$ is supported by $S(f)$; it remains to prove that if $f \leqslant f'$, $\{T, f\} = (f/f')\{T, f'\}$ or equivalently that, for any $\varphi \in \mathscr{C}^+(E)$, $T(\varphi f) = T((f/f') \cdot \varphi f')$, which is obviously true.

This projective family, indexed by V^+, will be called the *canonical projective family associated with T*.

We are going to prove that every projective family is, in some way, of that type.

PROPOSITION 25. *Let $(f_i, \mu_i)_{i \in I}$ be any projective family of elements of $\mathscr{D}^+(E) \times \mathscr{M}^+(E)$ and let V be the linear hereditary space of all $g \in \mathscr{D}^+(E)$ such that $|g| \leqslant nf_i$ for some i and some positive integer n.*

Then there exists a unique $T \in V_*^+$ such that, for every $i \in I$, $\mu_i = \{T, f_i\}$.

Proof. For any given i, let $V_i = \{\varphi f_i \colon \varphi \in \mathscr{C}(E)\}$; it is identical with the set of all $g \in \mathscr{D}(E)$ with $Sg \subset Sf_i$ and g/f_i bounded. Let us denote by T_i the positive form $g \to \mu_i(g/f_i)$ on V_i.

If there exists $T \in V_*^+$ such that $\mu_i = \{T, f_i\}$ for every i, its restriction to V_i must be identical with T_i; hence, as $V = \bigcup V_i$, T is unique.

Now, when $i \leqslant j$, $V_i \subset V_j$ holds and moreover, from $\mu_i = (f_i/f_j)\, \mu_j$ results also that, for any $g \in V_i$, $\mu_i(g/f_i) = \mu_j(g/f_j)$; hence $T_i = T_j$ on V_i. It follows that there is a (unique) linear form T on V with $T = T_i$ on V_i for each i; and T is $\geqslant 0$. By construction $\mu_i = \{T, f_i\}$ holds.

PROPOSITION 26. *Let us suppose V hereditary.*

(i) *For any $T \in V_*^+$, $f \in V^+$ and $g \in V$, the measure $\{T, f\}$ is supported by $S(f)$, the subquotient g/f is integrable with respect to $\{T, f\}$, and the submeasure $T_f = [f, \{f, T\}]$ is $\leqslant T$.*

(ii) *The mapping $f \to T_f$ is increasing on the directed set V^+ and its limit is T; more precisely, for any $g \in V$, $T(g) = T_f(g)$ for any $f \in V^+$ such that $g \leqslant nf$ for some integer n.*

Proof.

(i) We know already that $S(\{T, f\}) \subset S(f)$; what remains amounts to proving that for any $g \in V^+$, $\{T, f\}(g/f) \leqslant T(g)$.

Indeed $T_f(g) = \{T, f\}(g/f) = \sup_{\varphi f \leqslant g} \{T, f\}(\varphi) = \sup_{\varphi f \leqslant g} T(f\varphi) \leqslant T(g)$.

(ii) Let us notice that from these relations follows that $T_f(g) = \lim_{n \to \infty} T(\inf(g, nf))$. Hence, as the mapping $f \to \inf(g, nf)$ is increasing, the mapping $f \to T_f$ is also increasing. We know the rest already; in particular $T(g) = T_{|g|}(g)$ for any $g \in V$, hence $T = \lim_{f \in V^+} T_f$.

COROLLARY 27. *When V is hereditary, any extreme element $T \neq 0$ of V_*^+ (i.e., T on an extreme ray) is a submeasure.*

Proof. As $T \neq 0$, there exists an $f \in V^+$ with $T(f) \neq 0$; then $T_f \neq 0$ because $T_f(f) = T(f)$; and if T is extreme, T is proportional to T_f, hence it is a submeasure.

This statement will be clarified and extended to nonhereditary spaces V in Theorem 32.

PROPOSITION 28. *Let $T = [f_0, \epsilon_a]$ be any subvaluation on V, with $T \neq 0$.*

(i) *The set of all $x \in E$ such that $[f_0, \epsilon_x] = [f_0, \epsilon_a]$ is a closed set $K(f_0)$. For any $f \in V^+$ such that $T(f) = 1$, we have $T = [f, \epsilon_x]$ for any $x \in K(f_0)$. Hence, $K(f_0) \subset K(f)$ and $K(f)$ is independent of that $f \in V^+$ (and denoted K_T). For any $f, g \in V$, with $T(f) \neq 0$, the restriction of g/f to K_T is constant and equal to $T(g)/T(f)$.*

(ii) *If V is an algebra, all $g \in V$ are finite and constant on K_T; moreover, either $T = kT'$, where T' is a valuation, or all $g \in V$ are identically 0 on K_T and in this case T is degenerate multiplicative (i.e., $T(gh) = 0$ for any $g, h \in V$).*

We will omit the easy proof of this proposition.

IV. Extreme Positive Forms

Let us repeat that an element $T \in V_*^+$ will be called *extreme* when it is on some extreme ray.

For any V we will denote by $\hat{V}$ its hereditary completion, i.e., $\hat{V} = \{g \in \mathscr{D}(E): |g| \leqslant h$ for some $h \in V^+\}$. As a consequence of a well known theorem concerning the extension of positive forms, any $T \in V_*^+$ has at least an extension $\hat{T} \in \hat{V}_*^+$. When such an extension is unique for every T, we will say that V is *prehereditary*; in this case V_*^+ and $\hat{V}_*^+$ can be identified. Not all algebras are prehereditary (for instance polynomials on $\mathbb{R}$).

Let us say that V is *rich* if, for every $f \in V^+$, the set of all subquotients g/f bounded on $S(f)$ is uniformly everywhere dense in $\mathscr{C}(S(f))$. It is easy to check that *every rich space V is prehereditary*.

In the study of extreme positive forms, an important role will be played by measures $\{T, f\}$.

Lemma 29. *Let $T \neq 0$ be a submeasure on V, and for every $f \in \mathscr{D}^+(E)$ such that $T = [f, \pi]$ for some π, let $c(T, f)$ be the set of all $\mu \in \mathscr{M}^+(S(f))$ such that $T = [f, \mu]$; let us suppose, to simplify notations, that $T(f) = 1$.*

(i) *$c(T, f)$ is a convex compact subset of $\mathscr{M}^1(S(f))$; when V is hereditary, $c(T, f) = \{\pi\}$.*

(ii) *When T is extreme, T is a subvaluation, and $c(T, f) = \mathscr{M}^1(K_T)$, where K_T is the compact defined in* 28.*i*.

Proof.

(i) For any $g \in V^+$, the mapping $\mu \to \mu(g/f)$ of $\mathscr{M}^+(S(f))$ into $[0, \infty]$ is linear and continuous, so that the set of all μ for which $\mu(g/f) = T(g)$ is convex and closed. Hence $c(T, f)$ is also convex and closed, and as $\mu \in c(T, f)$ implies $\mu(1) = 1$, it is also compact. When V is hereditary, the relation $\mu(g/f) = \pi(g/f)$ implies that $\mu(\varphi) = \pi(\varphi)$ for all $\varphi \in \mathscr{C}(S(f))$, hence $\mu = \pi$.

(ii) Suppose T is extreme and let $\mu \in c(T, f)$; the relation $T(f) = 1$ implies $\| \mu \| = \mu(1) = 1$. For any $\mu' \leqslant \mu$ with $\mu' \neq 0$, $[f, \mu'] \leqslant [f, \mu]$ so that $[f, \mu'] = k[f, \mu]$, where $k = \| \mu' \|$ and hence $\| \mu' \|^{-1} \mu' \in c(T, f)$. As $c(T, f)$ is closed, this implies, after a well-known reasoning, that for any $x \in S(\mu)$, $\epsilon_x \in c(T, f)$. This proves that T is a subvaluation $[f, \epsilon_x]$ and that $S(\mu) \subset K_T$, so that $c(T, f) \subset \{\mu \in \mathscr{M}^+(K_T): \| \mu \| = 1\}$. Reciprocally, for any μ in this set, as for any $g \in V$, g/f is constant on K_T, $\mu(g/f) = T(g)$, hence $T = [f, \mu]$; this proves the given relation.

Remark 30. When T is a submeasure which is not extreme, $c(T, f)$ depends heavily on the choice of f in the relation $T = [f, \mu]$. The following example will show it. Let U be the linear space of real valued derivable functions on $A =$ (union of segments $[n, n + \frac{1}{2}]$ where $n = 0, 1, 2$), which are 0 at each of those n, with $f'(0) = f'(1) + f'(2)$. The relation $(fg)' = f'g + fg'$ proves that V is an algebra.

Let T be defined by $T(g) = g'(0)$; we have $T = [f_1, \epsilon_0] = [f_2, \epsilon_1 + \epsilon_2]$, where $f_1'(0) = 1$ and $f_1 = 0$ on $[1, 3/2] \cup [2, 5/2]$, and where $f_2 = 0$ on $[0, \frac{1}{2}]$ with $f_2'(1) = f_2'(2) = 1$. Here $c(T, f_1) = \{\epsilon_0\}$, whereas $c(T, f_2) = \{(\epsilon_1 + \epsilon_2)\}$.

LEMMA 31. *For any V and any $T \neq 0$ in V_*^+, there exists a proper submeasure $T' \leqslant T$ with $T' \neq 0$.*

Proof. Let $\hat{T}$ be a positive extension of T to the hereditary space $\hat{V}$. As $T \neq 0$, there exists some $f \in V^+$ with $T(f) = 1$; as also $\hat{T}(f) = T(f) = 1$, the submeasure $(\hat{T})_f = [f, \{\hat{T}, f\}]$ is $\neq 0$ and it is $\leqslant T$; its restriction T' to V is the answer.

THEOREM 32. *Let V be given, hereditary or not.*

(i) *Any extreme $T \in V_*^+$ is a proper subvaluation $[f, \epsilon_x]$ and the set of all associated poles x is the compact K_T (see 28).*

(ii) *The set of all subvaluations on V is closed in $V_*^+\{0\}$ (for $\sigma(V_*^+, V^+)$).*

(iii) *A subvaluation is not necessarily extreme in V_*^+, even if V is an algebra.*

(iv) *When V is an algebra, valuations on V are identical with multiplicative positive forms, and each of them is extreme in V_*^+.*

Proof.

(i) After Lemma 31, any extreme T in V_*^+ is a submeasure, and after 29.2, T is a subvaluation, whose pole can be any point in K_T.

(ii) For any $a \in E$, let us denote by ϵ_a the generalized valuation, with values in $\overline{\mathbb{R}}$, defined by $\epsilon_a(g) = g(a)$; and let us say that T in V_*^+ is a limit of products $k_i \epsilon_{a_i}$ if for every $g \in V$, $T(g) = \lim_{\mathscr{F}} k_i g(a_i)$, where $\mathscr{F}$ is a filter on the set of indices I. Any subvaluation $T = [f, \epsilon_a]$ is such a limit, for $T = \lim(f^{-1}(x)\, \epsilon_x)$, for $x \to a$ and $f(x) \neq 0$.

Reciprocally any such limit $T \neq 0$ is a subvaluation; indeed, if $T = \lim(k_i \epsilon_{a_i})$ according to some ultrafilter $\mathscr{U}$ on the set I of indices, we have for any $g \in V$: $T(g) = \lim_{\mathscr{U}} k_i g(a_i)$. Let a denote $\lim_{\mathscr{U}} a_i$. If $T \neq 0$, there exists $f \in V^+$ with $T(f) = 1$, hence $\lim_{\mathscr{U}} k_i f(a_i) = 1$.

So, for any $g \in V$, $T(g) = \lim_{\mathscr{U}} k_i g(a_i) = \lim_{\mathscr{U}} g(a_i)/f(a_i)$, hence $(g/f)(a) = T(g)$. In other words, $T(g) = [f, \epsilon_a]$.

It follows, after the double limit theorem, that any $T \in V_*^-$ which is a limit of subvaluations, is a limit of products $k_i \epsilon_{a_i}$, and so is itself a subvaluation.

(iii) In Remark 30, the linear space U is an algebra, and T defined by $T(g) = g'(0)$ is a subvaluation with pole 0, since $T(g) = [f, \epsilon_0]$ where $f(x) = x$ for every $x \in A$. However T is not extreme since $T = T_1 + T_2$, where $T_1(g) = g'(1)$, $T_2(g) = g'(2)$, and T_i is not proportional to T.

(iv) Obviously, if T is a valuation, T is positive and multiplicative; let us prove now that if T is multiplicative, T is extreme; this will prove (4) because if T is extreme, after 32(i) it is a subvaluation $[f, \epsilon_a]$, and if it is multiplicative, this subvaluation is a valuation; indeed: $T(f^2) = (T(f))^2 = ([f, \epsilon_a](f))^2 = 1$ and also $T(f^2) = (f^2/f)(a) = f(a)$, hence $f(a) = 1$ and it results that $[f, \epsilon_a] = [1, \epsilon_a]$.

So let I be a positive and multiplicative form on V; to prove that I is extreme, it is sufficient to prove that any submeasure less than I is proportional to I, because if this is true for any $T \leqslant I$, T is a limit of proper submeasures $T_i \leqslant T$ (see 26(ii)), and as every T_i is proportional to I, the same is true for T.

So let $T = [f, \mu] \leqslant I$, where $f \in V^+$ and $\mu \neq 0$.

(a) For any $f' \in V^+$ with $f \leqslant f'$, Proposition 20 shows that $T = [f', \mu']$, with $\mu' = (f'/f)\mu$. Let us show that on $S(\mu) = S(\mu')$, f' is equal to the constant $k = I(f')$: As $k \geqslant I(f) \geqslant T(f) = \mu(1) > 0$, we can put $f' = kg$. Then $I(g) = 1$ and as I is multiplicative, $I(g^n) = 1$ for any n. Now if we put $h = g - 2g^2 + g^3 = g(1-g)^2$, we have $I(h) = 0$ and also $T(h) \leqslant I(h)$, hence $T(h) = 0$; but as $T = [kg, \mu']$, we have $T(h) = k^{-1}\mu'(1-g)^2$; this implies $\mu'(1-g)^2 = 0$, hence on $S\mu'$ we have $g = 1$, that is $f' = k = I(f')$.

(b) It follows from this result that for any $f' \geqslant f$:

$$T(f') = [f, \mu](f') = \mu(f'/f) = \|\mu\|\, I(f')/I(f),$$

and by linearity this relation is true for any $f' \in V$; in other words $T = \alpha I$, where $\alpha = \|\mu\|/I(f)$.

This last result 32(iv) had been obtained previously by Bonsall, Lindenstrauss and Phelps (see [6]) in the case of algebras of functions everywhere defined and finite on an abstract set (in our present setting this corresponds to the case where the set of poles of valuations on V is everywhere dense in E).

COROLLARY 33 (concerning algebras). *Suppose V is an algebra containing* *some g_0 everywhere >0 (or equivalently, in view of abstract settings, some $g_0 \geqslant \alpha$, where the scalar α is >0.*

Then for any $T \in V_*^+$ with $T \neq 0$, the following statements are equivalent: (i) *T is extreme in V_*^+*; (ii) *T is a subvaluation*; (iii) *$T = kT'$ where T' is a valuation*; (iv) *$T = kT'$ where T' is multiplicative.*

Moreover the set $\mathscr{E}(V_*^+)$ of extreme elements is closed.

Proof. (i) $\Rightarrow$ (ii) results from 32(i).

(ii) $\Rightarrow$ (iii) because if $T = [f, \epsilon_a]$, $T(g_0) = (g_0/f)(a)$, hence as $g_0(a) > 0$ and $T(g) < \infty$, necessarily $f(a) \neq 0$ so that for any $h \in V$, $T(h) = (1/f(a))\,h(a)$, hence $T = f^{-1}(a)[1, \epsilon_a]$.

(iii) $\Rightarrow$ (iv) is obvious; and (iv) $\Rightarrow$ (i) results from 31(iv).

The final statement results from 31(ii) and the equivalence of 33(i) and 33(iii).

Remark 34. The set of degenerate multiplicative forms $T \in V_*^+$ (see 28), where V is an algebra, is obviously a (closed and hereditary) convex cone, so that such a T, is not in general a subvaluation.

Remark 35.

(i) There may exist multiplicative forms on an algebra V which are not positive. For instance, let V be the algebra of restrictions to $\mathbb{R}^+$ of real polynomials $p(x)$ on $\mathbb{R}$; then for any real $x < 0$, $g \to g(x)$ is a multiplicative form on V which is not a valuation and which is not positive. On the algebra V of real rational functions $r(x)$ on $\mathbb{R}$, there exist neither positive forms $T \neq 0$, nor multiplicative forms (use the fact that V is a field).

(ii) Here is an example of an algebra V on which there is no subvaluation $T \neq 0$ although V_*^+ is big: V is the algebra of continuous mappings of $[0, 1]$ into $\overline{\mathbb{R}}$, generated by 1 and the family of functions $\log |x - a|^{-1}$ with $a \in X$ where $X = [0, 1]$. This example could also be modified slightly so that there are subvaluations but no valuations.

(iii) Although the set of subvaluations is always closed in $V_*^+\backslash\{0\}$ (see 32(ii)), the set of their poles in E is not necessarily closed, even when V contains the constant function 1. Take for instance $E = \beta\mathbb{N}$ and $V = \mathscr{D}(E)$; every subvaluation T is extreme and of the form $k[1, \epsilon_x]$, hence the set of their poles is $\mathbb{N}$, which is not closed in E.

In fact the set of poles for a given V can be very bad, as in the following example (iv) which is interesting in several respects.

(iv) Let us consider a partition of $[0, 1]$ into two sets A, B, where A contains at least two points. And let V be the algebra defined in 35(ii), but with $X = A$. Each $g \in V$ is finite at each $b \in B$. The positive valuations on V are the linear forms T_b: $g \to g(b)$ where $b \in B$.

It can be shown (see for instance 55(i)) that every $T \in V_*^+$ is in fact a measure, in the sense that there exists a measure $\mu \in \mathscr{M}^+([0, 1])$ for which every $g \in V$ is integrable, with $T(g) = \mu(g)$. This implies that if $T \neq 0$, $T(1)$ also is $\neq 0$; so that $C = \{T \in V_*^+: T(1) = 1\}$ is a closed base of the weakly complete cone V_*^+; the set F of extreme points of C coincides, after 32(iv), with the set of valuations on V.

The bijection $b \to T_b$ of B onto F is bicontinuous because elements of the algebra V are continuous on [0, 1] and separate points of [0, 1]; hence F is homeomorphic with B, and so is as bad as B, although it is complete for the uniform structure of V_*^+.

When $B = [0, 1]$, this example proves that we cannot hope to get a theorem of integral representation in C, although V is an algebra containing 1, and although C satisfies a Krein–Milman property (C is the closed convex hull of F).

(v) Corollary 33 could lead to think that for any algebra V, the set $\mathscr{E}(V_*^+)$ of extreme elements of V_*^+ is closed. The example defined in Remark 30 proves that it is not true, even when V_*^+ has a closed base: indeed the subvaluations on V are, either those proportional to a valuation $[1, \epsilon_a]$, where $a \neq 0, 1, 2$, or those proportional to T_0, T_1, or T_2. All of them are extreme except those proportional to T_0; and as T_0 is a limit of extreme subvaluations $a^{-1}[1, \epsilon_a]$, $\mathscr{E}(V_*^+)$ is not closed.

(vi) $E = (\beta\mathbb{N}\setminus\mathbb{N})$ is an example of a substonian space such that the algebra $V = \mathscr{D}(E)$ admits many positive forms; the reason is that $\mathscr{D}(E) = \mathscr{C}(E)$. Indeed, every $f \in \mathscr{D}(E)$ has a continuous extension to $\beta\mathbb{N}$; and such an extension is necessarily bounded on $\mathbb{N}$ (hence also on E) because otherwise it would converge to infinity on some subsequence of $\mathbb{N}$, hence there would exist an open subspace of E on which it would be identically infinite.

(vii) Here is an example of a substonian space E, with $\mathscr{D}(F) \neq \mathscr{C}(F)$ for every open subset F of E, which admits valuations: Let A be the transfinite halfline $[0, \Omega]$, $\mathscr{T}$ the tribe generated by closed intervals $[0, \alpha]$, and $\mathscr{N}$ the σ-ideal of denumerable subsets of $[0, \Omega[$. Finally let $E =$ (space of maximal $\mathscr{T}$–$\mathscr{N}$-filters on A). On $\mathscr{D}(E)$ there is only one positive form (within a factor), the valuation $[1, \epsilon_\Omega]$.

THEOREM 36 (of Krein–Milman type). *Let V be given, and let P be the set of poles of valuations $T \neq 0$ (i.e., $P = \{x \in E: \forall f \in V, f(x)$ is finite and not identically 0$)$ and suppose $\bar{P} = E$, then*:

(i) *For every* $\mathcal{Q} \subset P$ *such that* $\bar{\mathcal{Q}} = E$, *the convex subcone of* V_*^+ *generated by the* $[1, \epsilon_x]$ *with* $x \in \mathcal{Q}$ *is everywhere dense in* V_*^+.

(ii) *Every subvaluation on* V *is a limit of elements* $k[1, \epsilon_x]$ *with* $x \in \mathcal{Q}$.

Proof.

(i) We have to prove that given $T \in V_*^+$, $f_1, ..., f_n \in V^+$ and $\epsilon > 0$, there is a measure $\pi > 0$ with finite support in $\mathcal{Q}$ such that $| T(f_i) - \pi(f_i)| < \epsilon$ for every i.

Let $f = \sum f_i$, $T_f = [f, \mu]$ (see 26(i)), $\varphi_i = (f_i/f) \in \mathscr{C}(S(f))$, and $\omega = \{x \in S(f): f(x) > 0\}$.

As $\bar{\omega} \supset S(\mu)$, there exists a discrete measure $\nu = \sum \beta_j \epsilon_{x_j}$ on $\mathcal{Q} \cap \omega$ such that $| \mu(\varphi_i) - \nu(\varphi_i) | < \epsilon$ for each i.

The measure $\pi = \sum \beta_j f^{-1}(x_j)\, \epsilon_{x_j}$ is the answer.

(ii) Use the same technique as for 32(ii).

Although every valuation $T \neq 0$ is not always extreme in V_*^+ this is true when V is an algebra (32(iv)); this justifies the qualification of "Krein–Milman type" for this Theorem.

EXAMPLE 37. The relation $\bar{P} = E$ holds in particular when the set of isolated points of E is everywhere dense in E. It is the case, for instance, when V, E are derived by a Stone–Cech procedure from a linear space U of functions everywhere defined and finite on an abstract set A. So we can assert that in such an abstract setting every T is a limit of positive linear combinations of valuations. We will give in Theorem 39 an interesting application of this remark to abstract algebras.

Remark 38.

(i) In 36(i), the set P of poles of valuations on V cannot be replaced by the set of poles of subvaluations: Let V be the linear subspace of $\mathscr{D}([0, 1])$ generated by $\mathscr{C}([0, 1])$ and functions $| x - a |^{-1/2}$ where $a \in [0, 1]$. For every a, $f \rightarrow \lim_{x \to a} | x - a |^{1/2} f(x)$ is a subvaluation with pole a. The set P of all those poles is everywhere dense in $[0, 1]$ and however the Lebesgue measure of $[0, 1]$, which is an element of V_*^+ is not a limit of positive linear combinations of those valuations. (As usual, this example can, of course, be translated in a substonian setting.)

(ii) Let us also remark that when V is an algebra and the set S of poles of subvaluations verifies $\bar{S} = E$, this implies $\bar{P} = E$: Indeed, 28(ii) shows that every $g \in V$ is identically zero on $\overline{(S \backslash P)}$; hence this set has an empty interior; hence, as $\bar{P} = E$, we have also $\bar{P} = E$.

THEOREM 39. *Let A be an abstract ordered algebra (on the reals) with $A = A^+ - A^+$ and $A^+ \cdot A^+ \subset A^+$; and let K denote the set of positive characters of A (i.e., positive multiplicative forms). The following statements are equivalent.*

(i) *The ordered algebra A is isomorphic with an algebra of real functions everywhere defined, and finite on a given set.*

(ii) a. *K separates the points of A;*
b. *A^+ is closed in A for the weak topology $\sigma(A, A_*^+)$;*
c. *the closed convex subcone of A_*^+ generated by K is identical with A_*^+.*

Proof. (i) $\Rightarrow$ (ii)a. trivially; (i) implies that A^+ is closed in A for $\sigma(A, K)$ hence also (ii)b.; (i) $\Rightarrow$ (ii)c. follows from 36 and 37.

Reciprocally, suppose (ii) is true; for every $f \in A$, let $\check{f}$ be the real function $m \to m(f)$ on K. The mapping $f \to \check{f}$ of A into the ordered algebra $\mathscr{F}(K, \mathbb{R})$ is positive, linear and multiplicative, because each $m \in K$ is a positive character; and (ii)a. shows that this mapping is injective.

In order to prove that it is an isomorphism of A onto $\check{A}$, it remains to show that $\check{f} \geqslant 0$ (i.e., $m(f) \geqslant 0$ for each m) implies $f \geqslant 0$; for this we need (ii)b. and (ii)c:

As A^+ is closed, A^+ is the polar of A_*^+ (duality theory), so that if the continuous form $x \to f(x)$ on A_*^+ is positive on P (which implies after (ii)c. that it is also positive on A_*^+), it belongs to A^+, hence $f \in A^+$.

Remark 40. Conditions (ii)a., b., and c. are independent. For instance, let A be the algebra of real functions on $\{0, 1\}$, and

$$A^+ = \{f \in A: 0 \leqslant f(0) \leqslant f(1)\}.$$

The algebra A is positively generated and $A^+ \cdot A^+ \subset A^+$; moreover (ii)a. and b. are satisfied, but not (ii)c., because the positive form $f \to (f(1) - f(0))$ is not a limit of positive linear combinations of valuations (which are here identical to positive characters).

The example which follows proves that (ii)a. and c. does not imply (ii)b.: Again A is the algebra of real functions on $\{0, 1\}$, and $A^+ = \{f \in A: (f = 0)$ or $(f(0) > 0$ and $f(1) > 0)\}$; again $A = A^+ - A^+$ and $A^+ \cdot A^+ \subset A^+$; moreover (ii)a., and c. are satisfied, but obviously not (ii)b.

Problem 41. There are several interesting problems concerning abstract ordered algebras on $\mathbb{R}$. For instance, Theorem 39 shows that *not* every positively generated algebra $V \subset \mathscr{D}(E)$ is order isomorphic with an algebra of functions defined and finite on a fixed set (for instance the algebra V of Lebesgue measurable functions on [0, 1]). But every such algebra has the interesting property Π that for any real polynomial $p(x_1, ..., x_n) \geqslant 0$ on $\mathbb{R}^n$, $p(f_1, ..., f_n)$ is positive in V for any choice of the f_i in V. If an abstract real algebra A has that property Π, when can it be represented in terms of subalgebras of some $\mathscr{D}(E)$? Remark 40 shows that *closedness* of A^+ should be required for that.

Let us notice that Π is stronger than asserting that $x^2 \geqslant 0$ for any $x \in A$; this is proved by the following example due to A. Connes.

Let A_n be the algebra of real polynomials $p(x_1, ..., x_n)$ and let A_n^+ be the set of sums of squares in A. It is an ordered algebra; but as there are positive polynomials (in the ordinary sense) which are not in A_n^+ (for an $n > 1$), property Π is not satisfied.

V. Structure of Positive Forms

We want now to get a better idea of the structure of any element $T \in V_*^+$. Let us begin at first with an arbitrary V.

Proposition 26(ii) and the theorem concerning extension of positive forms show already that any $T \in V_*^+$ is the limit of a directed set of submeasures $T_i \leqslant T$. This statement can be stated more precisely.

Theorem 42. *For any V, every $T \in V_*^+$ is the sum of a summable family of proper submeasures.*

Proof. It is a consequence of Lemma 31 and the theorem of Zorn. Let A be set of all sets X of proper submeasures on V, such that the sum T_X of all elements of X is $\leqslant T$. The set A, ordered by inclusion, is inductive, and so has maximal elements. Let X be such a maximal element; necessarily $T_X = T$; because otherwise there would exist, after Lemma 31 some proper submeasure $T' \leqslant (T - T_x)$ with $T' \neq 0$; the set $X' = X \cup \{T'\}$ would belong to A, hence a contradiction since X is maximal.

We will now study in greater detail the case where V is hereditary. All results given for hereditary V extend of course immediately to prehereditary V; this will not be repeated.

Lemma 43. *Suppose V is hereditary; then:*

(i) *V_*^+ is a complete lattice;*

(ii) *For any proper submeasure $[f, \pi] \neq 0$, the mapping $\alpha: \mu \to [f, \mu]$ of $\{\mu: 0 \leqslant \mu \leqslant \pi\}$ into $\{T \in V_*^+: T \leqslant [f, \pi]\}$ is a bijection;*

(iii) *for any proper submeasures $[f, \mu]$, $[f', \mu']$; $([f, \mu] = [f', \mu']) \Leftrightarrow (S(\mu) = S(\mu'); \mu' = (f/f')\mu; \mu = (f/f')\,\mu')$.*

Proof.

(i) V being hereditary is a lattice, hence V_*^+ is a complete lattice.

(ii) For any $g \in V^+$, $[f, \pi](g) = \sup\{[f, \pi](f\varphi): \varphi \in \mathscr{C}^+(E)$ and $f\varphi \leqslant g\}$. Hence for any $T \leqslant [f, \pi]$ and $(g - f\varphi) \geqslant 0$, the relation $T(g - f\varphi) \leqslant [f, \pi](g - f\varphi)$ proves that we have also $T(g) =$ (sup of all $T(f\varphi)$); so that if we put $\mu = \{T, f\}$ (see 24), for every $g \in V^+$, g/f is μ-integrable and $T = [f, \mu]$.

This proves that the mapping α is surjective; it is also injective because if $\mu_1 \neq \mu_2$, there exists $\varphi \in \mathscr{C}^+(E)$ with $\mu_1(\varphi) \neq \mu_2(\varphi)$, hence $[f, \mu_1](f\varphi) \neq [f, \mu_2](f\varphi)$.

(iii) Let us suppose $[f, \mu] = [f', \mu'] \neq 0$. If $S(\mu')\backslash S(\mu) \neq \varnothing$, as V is hereditary, there exists a $\varphi \in \mathscr{C}^+(E)$ which does not vanish identically on $S(\mu')$, with $S(\varphi)$ disjoint of $S(\mu)$, hence $[f, \mu](f\varphi) = 0$ and $[f', \mu'](f\varphi) = \mu'(\varphi) \neq 0$, a contradiction. Hence $S(\mu) = S(\mu') = K \subset S(f) \cap S(f')$.

For any $\varphi \in \mathscr{C}^+(E)$, $[f, \mu](f'\varphi) = \mu(\varphi \cdot f'/f) = \mu'(\varphi)$ and $[f', \mu'](f\varphi) = \mu(\varphi) = \mu'(\varphi \cdot f/f')$, hence $\mu' = (f'/f)\mu$ and $\mu = (f/f')\,\mu'$. The inverse implication is obvious.

COROLLARY 44. *If V is hereditary, every nonzero subvaluation on V has a unique pole, and is extreme in $V_*{}^+$.*

COROLLARY 45. *If V is hereditary, for any $f \in V^+$ and $T \in V_*{}^+$, T_f and $(T - T_f)$ are disjoint.*

Proof (of 45). Let $T' = \inf(T_f, (T - T_f))$, and remember that $T_f = [f, \{T, f\}]$. As $T' \leqslant T_f$, Lemma 43(ii) shows that $T' = [f, \mu']$ with $\mu' \leqslant \mu$. Moreover $(T - T_f)(f) = 0$, hence $T'(f) = 0$, hence $\mu'(1) = 0$ so that $\mu' = 0$.

COROLLARY 46. *If T is hereditary, every $T \in V_*{}^+$ is the sum of a summable family of mutually disjoint proper submeasures.*

Proof. Let A be the set of all sets X of proper submeasures on V, such that

(i) any two submeasures in X are disjoint;

(ii) the sum T_X of all elements of X is $\leqslant T$;

(iii) $(T - T_X)$ is disjoint of any element of X.

The set A ordered by inclusion is inductive; if X is a maximal element of A, $T = T_X$ because otherwise there exists an $f \in V^+$ with $(T - T_X)_f \neq 0$, and this submeasure, added to X, leads to a contradiction (as in 42).

EXAMPLE 47. Here is an example showing that if V is not hereditary, it may happen that 43(ii) is not verified. Let E_0 be a substonian space,

$V_0 \subset \mathscr{D}(E_0)$ and $T_0 \in (V_0)^*_+$, where T_0 is not a submeasure. Let E be the substonian space obtained from E_0 by addition of two distinct points $a, b \notin E_0$; and let $V = \{f \in \mathscr{D}(E): f_{E_0} \in V_0 \text{ and } f(a) = T_0(f_{E_0}) + f(b)\}$. Finally let us define T, T' on V by $T(f) = T_0(f_{E_0})$ and $T'(f) = f(a)$. Then $T \leqslant T'$ and T is not a submeasure although T' is a submeasure (and even a valuation).

Problem 48. When V is not hereditary, $V_*{}^+$ is generally not a lattice; however we can still say that two elements T_1, T_2 of $V_*{}^+$ are disjoint if there is no $T \neq 0$ in $V_*{}^+$ which is $\leqslant T_1$ and T_2. So we can still inquire whether, as in 46, any $T \in V_*{}^+$ is a sum of a family of proper submeasures, each of which disjoint from the sum of the others. The proof of 46 does not extend to this general problem.

We will now extend to positive forms on an hereditary V some of the notions which are known for measures.

DEFINITION 49. Suppose V is hereditary and let $T \in V_*{}^+$. The *support* of T is defined as the smallest closed subset $S(T)$ of E which supports all the measures $\{T, f\}$ for $f \in V^+$. Its complement is clearly the greatest open set ω such that $(S(f) \subset \omega)$ implies $(T(f) = 0)$.

A subset X of E is called *T-negligible* (resp. *T-measurable*) if it is negligible (resp. measurable) for each measure $\{T, f\}$.

A T-measurable real function θ on E is called a *T-factor* if it is positive and integrable for each measure $\{T, f\}$. For instance every bounded and positive borel function is a T-factor.

PROPOSITION 50. *Suppose V hereditary and $T \in V_*{}^+$. For each T-factor θ, the family of couples $(f, \theta\{T, f\})$ indexed by V^+ is projective* (*see* 23).

The positive form on V defined by this projective family (see 25) will be denoted by θT.

Proof. The mapping $f \to \theta\{T, f\}$ is increasing, and $\theta\{T, f\}$ is supported by $S(f)$. We have just to check that

$$(f \leqslant f') \Rightarrow (\theta\{T, f\} = (f/f') \cdot \theta\{T, f'\}.$$

This follows immediately from the projective character of the canonical family associated with T (see 24). Let us remark that for any $f \in V$, $(\theta T)(f) = \{T, f\}(\theta)$.

Application 51.

(i) For any T-measurable subset X of E, the *trace* of T on X is defined as $1_X T$. One can check that the traces of T on X and its complement X' are disjoint; and their sum is T. If $X = S(T)$, $1_X T = T$.

(ii) For any summable family (T_i) of elements of V_*^+ and any sequence (θ_n) of T-factors such that $\sum \theta_n$ is a T-factor, the following distributivity property holds: $(\sum \theta_n)(\sum T_i) = \sum \theta_n T_i$.

It should be noted here that when T is not a measure, the following distributivity property is false in general: $T(\sum f_n) = \sum T(f_n)$; the only relation is $\sum T(f_n) \leqslant T(\sum f_n)$, valid when $f_n \geqslant 0$.

We want to show now that multiplication by T-factors, although it is often useful, is not sufficient for some problems; in those cases, projective families (f_i, μ_i) are more useful.

DEFINITION 52. Suppose V hereditary and $T, T' \in V_*^+$. We will say that T' is *absolutely continuous* with respect to T if, for any $f \in V^+$, $\{T', f\}$ is absolutely continuous with respect to $\{T, f\}$.

This is equivalent to say that in the complete lattice V_*^+, T' belongs to the band generated by T. For instance, for any T-factor θ, θT is absolutely continuous with respect to T. One could hope that the converse be true; but we will shown by an example that it is false in general, even in the particular case when $T' \leqslant T$; this is due to the fact that the set of measures $\{T, f\}$ associated with a given T cannot be replaced in general by a denumerable set.

PROPOSITION 53. *Suppose that V is hereditary and has a denumerable cofinal subset. Then any T' absolutely continuous with respect to T can be written $T' = \theta T$, where θ is a uniquely determined (up to T-negligible subsets) T-factor.*

Proof. Let (f_n) be an increasing sequence in V^+, cofinal with V^+. For each n, there is an (essentially) unique function $\theta_n \geqslant 0$ on E, integrable with respect to $\{T, f_n\}$ such that $\{T', f_n\} = \theta_n\{T, f_n\}$. One can check that for any p, q with $p \leqslant q$, $(\theta_p - \theta_q)$ is negligible with respect to $\{T, f_p\}$. It follows from a known result on measure theory that there exists a function $\theta \geqslant 0$ on E, integrable with respect to each $\{T, f_n\}$, and such that for each n, $(\theta - \theta_n)$ is $\{T, f_n\}$-negligible. This is the T-factor we were looking for.

This proof could be slightly modified to make Proposition 53 appear as a particular case of the following simple result. Let $T, T' \in V_*^+$ such that $T' = \sum T_n'$, with $T_n' = \theta_n T$, where θ_n is a T-factor. Then $T' = \theta T$ where $\theta = \sum \theta_n$.

General Case and Counter-Example 54. In the general case, the problem amounts to the following: suppose V hereditary and T given $\in V_*^+$. Let (f_i) be a cofinal directed subset of V^+, $\mu_i = \{T, f_i\}$ and $\theta_i \in L_1^+(\mu_i)$ such that whenever $i \leqslant j$, $(\theta_i - \theta_j)$ is μ_i-negligible. Then, does there exist a T-factor θ such that for every i, $(\theta - \theta_i)$ is μ_i-negligible?

The following example will show that it may be false, even when the θ_i are $\leqslant 1$; in other words, when V has no denumerable cofinal subset, the Radon–Nikodym theorem is not valid in general in V_*^+; here is the example.

Let $\mathscr{T}$, $\mathscr{N}$ be, respectively, the tribe of measurable subsets and the set of negligible subsets associated to the Lebesgue measure λ on $I = [0, 1]$. Let us call U the linear space of elements $f \in M(\mathscr{T}, \mathscr{N})$ (see 13 and 14) such that $|f|$ is bounded by some finite sum of functions $|x - a|^{-1/2}$, where $a \in [0, 1]$. Let $E = \tilde{I}(\mathscr{T}, \mathscr{N})$ and V the image of U into $\mathscr{D}(E)$. For every $a \in [0, 1]$, let $\mathscr{C}_a$ be the set of all maximal $(\mathscr{T} - \mathscr{N})$-filters converging to a; in each $\mathscr{C}_a$, let us choose one element r_a, and let $A = \{r_a\colon a \in [0, 1]\}$; finally let $\tilde{\lambda}$ be the image of λ on E. We define the elements T_1, T_2 of V_*^+ by:

$$T_1(f) = \tilde{\lambda}(f); \qquad T_2(f) = \sum_{x \in A} (\lim_{y \to x} |y - x|^{1/2} f(y)).$$

Finally, let $T = T_1 + T_2$; we have $T_1 \leqslant T$ and however we will show that there is no T-factor θ such that $T_1 = \theta T$.

Indeed, such a θ must vanish identically on A, and however $(1 - \theta)$ must be $\tilde{\lambda}$-negligeable; in order to get a contradiction, it is sufficient to choose A with an exterior $\tilde{\lambda}$-measure equal to 1. Now, as E is stonian, this is equivalent to saying that $\bar{A} = E$; and with the axiom of choice it is easy to choose A with this property: Use the fact that [0, 1] and the set of clopen subsets of E have the same power, and determine the r_a by transfinite induction.

Let us come back now to the study of linear spaces which are not necessarily hereditary.

Two particular Cases 55.

(i) Suppose that V is *adapted with respect to some* $u \in V^+$, in the sense that: $\forall f \in V^+$, $\exists g \in V^+$, $\forall \epsilon > 0$, $\exists k \in \mathbb{R}^+$ such that $f \leqslant ku + \epsilon g$. For any $T \in V_*^+$, let us extend T (into $\hat{T}$) to the hereditary completion $\hat{V}$ of V; then again $\hat{V}$ is adapted with respect to u. *Let us prove that* $\hat{T} = \hat{T}_u$, *which will prove that* $T = [u, \mu]$ *for some measure* μ. Indeed, if $T' = \hat{T} - \hat{T}_u$, we notice that $T'(u) = 0$; hence the relation $f \leqslant ku + \epsilon g$ implies $T'(f) \leqslant \epsilon T'(g)$, where ϵ is arbitrary; hence $T'(f) = 0$ for any $f \in V^+$, so that $T' = 0$.

This adaptation property with respect to u is verified when V has a unit order u (i.e., every $g \in V^+$ is less than some ku) or when V is an algebra which contains an element $u > 0$ everywhere on E; in this last case every $T \in V_*^+$ can be in fact identified with a measure on E.

(ii) Suppose now that V is *adapted with respect to constants* (these constants not belonging necessarily to V), in the sense that: $\forall f \in V^+$, $\exists g \in V^+$, $\forall \epsilon > 0$, $\exists k \in \mathbb{R}^+$ such that $f \leqslant k + \epsilon g$.

This is true for instance when each f in V is bounded or when V is an algebra. Let Z be the closed set of common zeros $\{x \in E: \forall f \in V, f(x) = 0\}$ and let $\Omega = \complement Z$. And for simplicity sake, let us suppose V hereditary (if not, use extensions of elements T to $\hat{V}$). Then (see 51) for every $T \in V_*^+$, $T = 1_Z \cdot T + 1_\Omega \cdot T$. Let us show that $1_\Omega \cdot T$, *the restriction of T to Ω, is a Radon measure μ on the locally compact space Ω*, in the sense that for every $f \in V$, $(1_\Omega \cdot T)(f) = \mu(f)$.

Let $\mathscr{K}(\Omega)$ be the set of all $g \in \mathscr{C}(E)$ with $S(g) \subset \Omega$. As $Z \cap \Omega = \varnothing$, $\mathscr{K}(\Omega) \subset V$; hence the linear form $g \to (1_\Omega T)(g)$ is defined on $\mathscr{K}(\Omega)$ and is a positive measure μ (not necessarily bounded) on Ω. Now, for every $f \in V^+$,

$$(1_\Omega T)(f) = (1_\Omega\{T, f\})(1) = \sup\{T(\varphi f): \varphi \in \mathscr{K}(\Omega), \varphi \leqslant 1\}.$$

On the other hand:

$$\mu(f) = \sup\{(1_\Omega T)(g): g \in \mathscr{K}(\Omega), g \leqslant f\} = \sup\{T(g): g \in \mathscr{K}(\Omega), g \leqslant f\}.$$

Obviously $\mu(f) \leqslant 1_\Omega T$, so that, after replacing T by $(T - \mu)$, one can suppose finally that $\mu = 0$, i.e. $T(\varphi) = 0$ for every $\varphi \in \mathscr{K}(\Omega)$. We have to prove that $1_\Omega T = 0$, i.e. $T(\varphi f) = 0$ for any $\varphi \in \mathscr{K}(\Omega)$. From the relation $f \leqslant k + \epsilon g$, one gets $\varphi f \leqslant k\varphi + \epsilon \varphi g$, hence $T(\varphi f) \leqslant \epsilon T(\varphi f)$, hence $T(\varphi f) = 0$ since ϵ is arbitrary.

A generalization 56. We want now to say a few words concerning spaces $M(\mathscr{T}, \mathscr{N})$ of measurable functions when $\mathscr{N}$ is not a σ-ideal (see 9–14).

Let U be a positively generated linear subspace of some space $M(\mathscr{T}, \mathscr{N})$ where the ideal $\mathscr{N}$ of $\mathscr{T}$ is arbitrary. To study positive forms T on U is equivalent to studying positive forms T' on the preimage U' of U in $M(\mathscr{T})$, such that $T'(f') = 0$ whenever f' is $\mathscr{N}$-negligible. The interpretation of this situation in the substonian space $E = \tilde{I}(\mathscr{T})$ through the mapping $f \to \tilde{f}$ of $M(\mathscr{T})$ into $\mathscr{D}(\tilde{I}(\mathscr{T}))$ is easy: Let Ω be the union of all clopen subsets of E of the form $\tilde{X}$, where $X \in \mathscr{N}$. Let us suppose also (which is always possible after an extension) that U is hereditary. The mapping $T \to \tilde{T}$ defined by $\tilde{T}(\tilde{f}) = T'(f') = T(f)$ for any f' in the class f, is an isomorphism of U_*^+ onto the subcone of elements $\tau \in V_*^+$ such that $S(\tau)$ (defined in 49) is contained in $(E \backslash \Omega)$.

So, finally the study of U_*^+ is equivalent to the study of the elements T of some V_*^+ (where V is hereditary), such that $S(T)$ is contained in a given closed set F of E.

Most of the theorems concerning V_*^+ are still valid; for instance Theorem 32 concerning extreme elements is valid; and to make Theorem 36 valid it is sufficient to replace condition $\bar{P} = E$ by $\bar{P} = F$.

VI. Order Saturation of an Hereditary Space V

We want to prove that on an hereditary space V, all positive forms have an extension to a canonical bigger space $\overline{V}$ containing V.

Lemma 57. *Let U be a Riesz linear space, and let V be an hereditary (or solid) linear subspace of U. For every $T \in V_*^+$, the mapping $\overline{T}$ of U^+ into $[0, \infty]$ defined by $\overline{T}(x) = \sup\{T(y): y \in V^+, y \leqslant x\}$ is positively linear. Moreover $\overline{T} \leqslant T'$ on U^+ for any positive linear extension T' of T to U.*

Proof. As $\overline{T}$ is increasing on U^+ it is sufficient to prove that $\overline{T}$ is additive on U^+: for any $a, b, z \in U^+$ with $z \leqslant a + b$, there exist $x, y \in U^+$ with $x \leqslant a$, $y \leqslant b$ and $x + y = z$; it follows that:

$$\begin{aligned}\overline{T}(a+b) &= \operatorname{Sup}\{T(z): z \in V \text{ and } z \leqslant a + b\}\\ &= \sup\{T(x): x \in V \text{ and } x \leqslant a\} + \sup\{T(y): y \in V \text{ and } y \leqslant b\}\\ &= \overline{T}(a) + \overline{T}(b).\end{aligned}$$

Moreover

$$\overline{T}(a) = \operatorname{Sup}\{T(x): x \in V \text{ and } x \leqslant a\} = \operatorname{Sup}\{T'(x): x \in V \text{ and } x \leqslant a\} \leqslant T'(a).$$

Theorem 58. *Let be given V hereditary $\subset \mathscr{D}(E)$ and $T \in V_*^+$. The set of all positively generated linear subspaces of $\mathscr{D}(E)$ on which T admits a positive linear extension has a greatest element $\overline{V}_T = \{f \in \mathscr{D}(E): \overline{T}(|f|) < \infty\}$.*

This space is hereditary and the positive form on $\overline{V}_T$ which is equal to $\overline{T}$ on $(\overline{V}_T)^+$ is the smallest positive extension of T to this space.

Proof. As $\overline{T}$ is positively linear and increasing on $\mathscr{D}^+(E)$, $\overline{V}_T$ is a linear hereditary space.

For any positively generated subspace U of $\mathscr{D}(E)$, any positive linear extension T' of T to U can be also extended to $\hat{U}$; so let us suppose U is hereditary (and hence a Riesz space); Lemma 57 proves that $\overline{T} \leqslant T$ on U^+; it follows that $\overline{T} < \infty$ on U^+, hence also $U \subset \overline{V}_T$.

Definition 59. The intersection $\overline{V}$ of all $\overline{V}_T$ (for $T \in V_*^+$) is called the *order saturated* space of V.

$\overline{V}$ is the greatest possible hereditary space of $\mathscr{D}(E)$ on which every $T \in V_*^+$ can be positively extended. Obviously $\overline{\overline{V}} = \overline{V}$. For instance every space $L^p(\mu)$ (for $p \geqslant 1$) is order-saturated in the space of all μ-measurable functions.

Problem 60. For a given hereditary $V \subset \mathscr{D}(E)$, let V' be the set of all $g \in \mathscr{D}(E)$ such that g/f is μ-integrable for every proper submeasure $[f, \mu]$. Obviously $\overline{V} \subset V'$; are those spaces identical?

VII. Application to the Study of Weakly Complete Convex Cones

We will denote by $\mathscr{S}$ the class of all proper weakly complete convex cones. For any $X \in \mathscr{S}$, let us denote by X^0 the cone of continuous positive forms on X; the theory of duality shows that X is nothing else than $(X^0)_*^+$. As X^0 is an ordered cone of real valued functions, this proves that the class of our cones V_*^+ is in fact identical with $\mathscr{S}$. Hence we must expect that the notions studied in this work can be useful for the study of $\mathscr{S}$.

As an illustration we will give here an application of the use of ultrafilters.

Example 61. This example was devised to answer a question of M. Rogalski concerning universally well-caped cones.

Let $\mathscr{U}$ be a non trivial ultrafilter on the set $\mathbb{N}$ of positive integers, and let $u \in \mathscr{C}_0{}^+(\mathbb{N})$ with $u > 0$ and chosen once for all. Let $V_{\mathscr{U}} = \{f \in \mathscr{C}_0(\mathbb{N}) : \exists X_f \in \mathscr{U}$ with $f_{X_f} = o(u)\}$.

The linear space $V_{\mathscr{U}}$ is hereditary; moreover for any $f \in V_{\mathscr{U}}{}^+$ there exists another $g \in V_{\mathscr{U}}{}^+$ with $g > 0$ such that $\lim_{n\to\infty}(f/g)(n) = 0$: Take $g = [f^{1/2} + u]$ on the complement of X_f; and $g = [u^2 + (uf)^{1/2}]$ on X_f. This implies that $V_{\mathscr{U}}$ is an adapted subspace of $\mathscr{C}(\mathbb{N})$ and hence (see Choquet 1) every positive form on $V_{\mathscr{U}}$ is a measure μ on $\mathbb{N}$. This measure is bounded; otherwise there would exist a partition of $\mathbb{N}$ into A_1, A_2 with $\mu(A_1) = \mu(A_2) = \infty$; as $\mathscr{U}$ is an ultrafilter, A_1 or A_2 does not belong to $\mathscr{U}$, and so the restriction of $V_{\mathscr{U}}$ on this A_i is exactly $\mathscr{C}_0(A_i)$; hence a contradiction since $\mu(A_i) = \infty$.

It results that $(V_{\mathscr{U}})_*^+ = l_1{}^+$.

Let $\mathscr{T}_{\mathscr{U}}$ be the topology induced by $\sigma(l_1, V_{\mathscr{U}})$ on $l_1{}^+$; as $\mathscr{T}_{\mathscr{U}}$ is Hausdorff and weaker than the restriction of $\sigma(l_1, \mathscr{C}_0(\mathbb{N}))$ to $l_1{}^+$, those two topologies have identical restrictions to the universal cap $K = \{\mu \in l_1{}^+ : \|\mu\| \leq 1\}$ of $l_+{}^1$.

The cone $l_1{}^+$ is weakly complete for all weak topologies $\sigma(l_1, V_{\mathscr{U}})$. Let us show that all topologies $\mathscr{T}_{\mathscr{U}}$ are different and even not comparable; it is sufficient for that, to show that if $\mathscr{U}_1 \neq \mathscr{U}_2$ the bases of neighborhoods of 0 in $l_1{}^+$ are not comparable in $\mathscr{T}_{\mathscr{U}_1}$ and $\mathscr{T}_{\mathscr{U}_2}$.

As $\mathscr{U}_1 \neq \mathscr{U}_2$ there is a partition of $\mathbb{N}$ into X_1, X_2 supporting $\mathscr{U}_1$, $\mathscr{U}_2$, respectively. Now, for each $\mathscr{U}$, as $l_1{}^+$ is weakly complete for $\sigma(l_1, V_{\mathscr{U}})$, the set of slices $\mathscr{J}_g = \{\mu : \mu(g) \leq 1\}$ for $g \in V_{\mathscr{U}}{}^+$ is a base of neighborhoods of 0 in $\mathscr{T}_{\mathscr{U}}$.

Let us take $g_1 \in V_{\mathscr{U}_1}$ such that $g_1 = u$ on X_2; the slice $\mathscr{J}(g_1)$ cannot contain any slice $\mathscr{J}(g_2)$ where $g_2 \in V_{\mathscr{U}_2}$ because, as $g_2 = o(u) = o(g_1)$ on an infinite subset of X_2, there exists some $\mu \in l_1{}^+$ which is supported by X_2, with $\mu(g_2) = 1$ and $\mu(g_1) = \infty$. And vice-versa; hence $\mathscr{T}_{\mathscr{U}_1}$, $\mathscr{T}_{\mathscr{U}_2}$ are not comparable. And so, briefly:

Theorem 62. *There exists on l_1 a family of 2^c weak topologies, strictly* *weaker than $\sigma(l_1, \mathscr{C}_0(\mathbb{N}))$, whose restrictions to l_+^1 are mutually noncomparable topologies and whose restrictions to the universal cap K are identical.*

Let us point out that for any finite family $(\mathscr{U}_i)$ of nontrivial ultrafilters on $\mathbb{N}$, l_1^+ is again complete for the weak topology $\sigma(l_1, \bigcap_i V_{\mathscr{U}_i})$. This proves that $(l_1^+, \sigma(l_1, V_{\mathscr{U}}))$ is not minimal, in the sense of Choquet [2] for any choice of $\mathscr{U}$.[2]

EXAMPLE 63. A slightly different example is obtained by taking

$$V_{\mathscr{U}}' = \{f \in \mathscr{C}_0(\mathbb{N}) \colon \lim_{\mathscr{U}} g/u = 0\}.$$

This space is not adapted, but by a longer argument, it can be proved that $(V_{\mathscr{U}}')_*^+ = l_1^+$ and that the corresponding topologies $\mathscr{T}_{\mathscr{U}}'$ again are noncomparable.[3]

VIII. HISTORICAL BACKGROUND AND A PROBLEM

Remark 64. Notions more or less linked to those which we have studied: measures $\{T, f\}$, products θT, can be found in previous work.

The spectral families of measures $\mu_{x,x'}$ studied by D. A. Edwards and C. T. Ionescu Tulcea in [5] look like our families of measures $\{T, f\}$; but their theory is based on Banach spaces instead of ordered spaces, and moreover the aim is the study of algebras of operators, not the study of positive forms.

Problem 65. Let us end this paper with a problem. Let U be the space of continuous real functions f on $\mathbb{R}^+$ such that

$$I(f) = \limsup_{a \to \infty} \left(a^{-1} \int_0^a |f|^p(t)\, dt\right) < \infty,$$

and let $\|f\| = (I(f))^{1/p}$ (where $p \geqslant 1$). The quotient of U by the subspace $\{f \in U : I(f) = 0\}$ is a Banach space V, called the Marcinkiewicz space of order p. It has an obvious order, for which it is lattice, and for which every interval

[2] An example of a minimal $X \in \mathscr{S}$ which is not locally compact is given by $\mathbb{R}_+^N$ with the topology $\sigma(\mathbb{R}^N, \mathbb{R}^{(N)})$.

[3] In that trend of ideas, let us point out the following useful property. Let H_1, H_2 be two positively generated subspaces of $\mathscr{C}_0(\mathbb{N})$ with $(H_i)_*{}^+ = l_1^+$. Then the identity of traces of $\sigma(l_1, H_1)$ and $\sigma(l_2, H_2)$ on l_1^+ is equivalent to the fact that any $f \in H_1^+$ is less than some $g \in H_2^+$, and reciprocally [see 2, p. 175].

is bounded. Every positive form on V is continuous and every continuous form is the difference of two positive forms, hence the interest of determining positive forms.

The convex cone V_*^+ is the direct sum of two closed faces P_s, P_r, where P_s is the set of positive forms T such that $T(1) = 0$, and P_r is the set of positive forms T such that $T(f) = 0$ for every positive singular f (i.e., $\inf(f, 1) = 0$). The cone V_*^+ has *no extreme element*. The elements of P_r (but not of P_s) can be represented canonically as Radon measures on a closed subset K of $(\beta\mathbb{R}^+ \setminus \mathbb{R}^+)$. It would be interesting to prove that each element T of P_r (resp. P_s) is a "mixture" of elementary positive forms $T_{\alpha,\mathscr{U}}$, where $\alpha \in V$, with $\mathscr{U}$ an ultrafilter on $\mathbb{R}^+$ converging to $+\infty$, and

$$T_{\alpha,\mathscr{U}}(f) = \lim_{\mathscr{U}} \left(a^{-1} \int_0^a \alpha(t) f(t)\, dt\right).$$

Complements 66. Since this lecture (Oct. 70), some new developments were obtained: (1) A systematic study of subspaces of $\mathscr{C}_0(\mathbb{N})$ associated with an ultrafilter on $\mathbb{N}$; (2) A useful canonical decomposition of every $T \in V_*^+$ (where V is hereditary) into three parts: T_0, T_m, T_∞, where T_0 is supported by Z, the set of common zeros of V, T_m is a measure on $(E \setminus Z)$, and T_∞ vanishes on bounded elements of V.

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